Verifying Differentiability of Solutions of Nonsmooth Dynamic Systems

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Nonsmoothness can enter dynamic systems through phenomena such as thermodynamic phase changes and flow regime transitions. We consider the broad class of nonsmooth dynamic systems that can be represented as solutions of parametric ordinary differential equations (ODEs) with right-hand side (RHS) functions that are finite compositions of analytic functions and absolute-value functions. As shown in [2], state variables of these systems may be nondifferentiable with respect to system parameters. In this case, when using optimization methods which require first-order sensitivity information, generalized derivatives [1,2] must be computed. These generalized derivatives are significantly more difficult to furnish than the classical derivative.

Nevertheless, Clarke [1] shows that it is possible for state variables to be differentiable with respect to system parameters, even when nonsmooth domain points of the RHS function are visited along an ODE solution trajectory. Moreover, Clarke provides a sufficient condition for this differentiability: namely, that nonsmooth domain points of the RHS function be reached only at a set of times with zero Lebesgue measure.

By exploiting the structure of the considered ODE RHS functions, we show that Clarke's sufficient condition for differentiability can be recast in a numerically tractable form [3]. Firstly, we show that Clarke's sufficient condition is satisfied if there is no "valley-tracing mode", which is a nonzero duration on which the argument of some absolute-value function in the RHS is identically zero. Secondly, we show that any valley-tracing mode must begin either at the initial time, or when some other absolute-value function in the RHS switches between its two linear pieces. This condition can be detected numerically using standard event-detection techniques [5], and can sometimes be shown *a priori* never to hold. If no valley-tracing mode is detected for a particular parameter value, then a classical derivative exists, and may be computed using classical ODE sensitivity theory. Numerical examples are presented for illustration, in which nonsmooth flow systems are optimized [4].

[1] F. H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley & Sons, Inc., New York, 1983.

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[4] K. A. Khan, A. M. Sahlodin, M. Yunt, and P. I. Barton, Parametric sensitivity analysis of dynamic systems using nonsmooth analysis: II. Computation. In preparation (2014).

[5] T. Park and P. I. Barton, State event location in differential-algebraic models, ACM T. Model. Comput. S., 6 (1996) 137-165.